

# Positivity properties of relative complete intersections

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## Abstract

We give conditions for  $f$ -positivity of relative complete intersections in projective bundles. We also derive an instability result for the fibres.

## 1 Introduction

Let us consider a fibred  $n$ -dimensional variety  $f: X \rightarrow B$  over a smooth curve  $B$ , and a line bundle  $\mathcal{L}$  on  $X$ . We say that  $\mathcal{L}$  is  $f$ -positive if the following inequality holds (cf. [1, Def. 1.3])

$$L^n \geq n \frac{L_{|F}^{n-1}}{h^0(F, L_{|F})} \deg f_* \mathcal{L}. \quad (1.1)$$

This property encodes natural inequalities relating invariants of  $X$ ,  $B$  and the general fibre  $F$ . It has a profound relation with stability properties of the pair  $(F, \mathcal{L}_F)$ . The best understood situation happens for  $n=2$ . In this case, several stability properties of the general fibres are equivalent and  $f$ -positivity can be proven for a large class of line bundles. The case  $\mathcal{L} = K_f$  is of particular interest, and it gives the so called *slope inequality* for fibred surfaces (see [1] for a complete discussion).

For higher values of  $n$  only few results are known. In [2] the authors study the case where  $X$  is a relative hypersurface in a projective bundle  $\mathbb{P}_B(\mathcal{E})$ , where  $\mathcal{E}$  is a vector bundle over a curve  $B$ . There  $f$ -positivity is completely understood by a numerical relation among the class of  $X$  in the Néron-Severi space  $N^1(\mathbb{P})$  and the slope of  $\mathcal{E}$ . From this we can deduce instability and singularity conditions for the fibres and also for the total space  $X$ .

In the present paper we study  $f$ -positivity for relative complete intersections  $X$  in a projective bundle  $\mathbb{P} := \mathbb{P}_B(\mathcal{E})$ , and extend to this set-up as far as possible the results in [2].

First of all we find a necessary and sufficient numerical condition for  $f$ -positivity of  $\mathcal{O}_X(1)$  and its powers, depending on the class of  $X$  in the Néron-Severi space of cycles  $N^c(\mathbb{P})$  (Theorem 2.1). Indeed the  $f$ -positivity of low powers of  $\mathcal{O}_X(1)$ , as well as the  $f$ -positivity of its high enough powers, turns out to be equivalent to a simple inequality relating the numerical class of  $X$  in  $N^c(\mathbb{P})$  with the slope of  $\mathcal{E}$ . This is a generalization of Theorem 2.8 of [2], where we have a condition for *any* power of  $\mathcal{O}_X(1)$  (see Remark 2.7).

We then observe that the condition obtained defines a cone  $\mathbb{B}$  in the 2-dimensional space  $N^c(\mathbb{P})$  which determines the classes of  $f$ -positive complete intersections  $X$  (Proposition 3.6). By reformulating in a suitable way a result of Fulger [5], using the so-called virtual slopes of the vector bundle  $\mathcal{E}$ , we see that the cone  $\mathbb{B}$  is always intermediate between the Pseudoeffective and the Nef cones in  $N^c(\mathbb{P})$ .

In the last section we turn our attention to the Chow stability of the general fibres. From a result of Ferretti we see that the complete intersection of semistable varieties is semistable (Proposition 4.2). Thus by combining this with a result of Bost [3] and a result of Lee [6] we can prove a general result: the line bundle  $\mathcal{O}_X(h)$  is  $f$ -positive for fibrations whose general fibre is a complete intersection of hypersurfaces with mild singularities (Theorem 4.5 and Remark 4.7).

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Eventually, using the same reasoning as in [2], we combine the result of Bost with Theorem 2.1 proved in the first part of the paper and prove an unstability condition for the fibres of relative complete intersections: Theorem 4.8.

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## 2 Inequalities for invariants of relative cycles

We work over the complex field. Let  $\mathcal{E}$  be a vector bundle of rank  $r \geq 3$  and degree  $d$  on a smooth projective curve  $B$  of genus  $b$ . Consider the relative projective bundle  $\mathbb{P} := \mathbb{P}_B(\mathcal{E})$  with its projection  $\pi: \mathbb{P}_B(\mathcal{E}) \rightarrow B$ . Let  $\mathcal{O}_{\mathbb{P}}(1)$  be the tautological sheaf on  $\mathbb{P}$ .

Let  $X_1, \dots, X_c \subset \mathbb{P}$  be relative hypersurfaces: each  $X_i$  is an effective divisor in a linear system of the form  $[k_i H - \pi^* M_i]$ , where  $k_i > 1$  and  $M_i$  is a divisor on  $B$ , say of degree  $y_i \in \mathbb{Z}$ .

Let  $X$  be the scheme theoretic intersection of the  $X_i$ 's, and  $f: X \rightarrow B$  the induced fibration. We assume that the intersection  $X$  is irreducible and proper, i.e. of dimension  $r - c$ .

Consider the sheaf  $\mathcal{O}_X(1) = j^* \mathcal{O}_{\mathbb{P}}(1)$  on  $X$ , where  $j$  is the natural inclusion  $j: X \rightarrow \mathbb{P}$ . In this section we study the  $f$ -positivity of  $\mathcal{O}_X(1)$  and of its powers. Similarly to the case of codimension 1 treated in [2], this property turns out to be in many cases equivalent to a simple inequality between the slopes  $y_i/k_i$  and the slope of the sheaf  $\mu = \mu(\mathcal{E}) = d/r$ . However, while for codimension 1 we have a result holding for all positive multiples of  $\mathcal{O}_X(1)$ , in bigger codimension we can completely treat only the case of low powers and of high enough powers: see Remark 2.7. The main result is the following:

**Theorem 2.1.** *With the notations above, the following statements are equivalent*

- (1)  $\sum_{i=1}^c \frac{y_i}{k_i} \leq c\mu$ ;
- (2) the line bundle  $\mathcal{O}_X(1)$  is  $f$ -positive;
- (3) the line bundle  $\mathcal{O}_X(h)$  is  $f$ -positive for any  $h < \min_i \{k_i\}$ ;
- (4) the line bundle  $\mathcal{O}_X(1)$  is asymptotically  $f$ -positive, i.e.  $\mathcal{O}_X(h)$  is  $f$ -positive for  $h \gg 0$ .

**Remark 2.2.** This result in particular implies that  $f$ -positivity for complete intersections is stable under intersection. Indeed, it is immediate to derive from it that if two relative complete intersections  $Y$  and  $Z$  in  $\mathbb{P}$  intersecting properly are  $f$ -positive, then their intersection is  $f$ -positive. This property should be compared with an analogue result holding for Chow stability (Proposition 4.2).

The theorem above follows by making explicit inequality (1.1) in our setting. Let us first compute the invariants associated to the  $f$ -positivity of  $\mathcal{O}_X(h)$ . The longest calculation is for  $\deg f_* \mathcal{O}_X(h)$ ; in order to describe it, let us introduce some notation. For a multi-index  $I = \{i_1, i_2, \dots, i_l\} \subseteq \{1, 2, \dots, c\}$ , we call  $|I| = l$  its length. We shall indicate by  $k_I$  and  $y_I$  the corresponding sums:

$$k_I = \sum_{j=1}^l k_{i_j}, \quad y_I = \sum_{j=1}^l y_{i_j}.$$

Moreover, for  $I = \emptyset$ , we define  $k_I = y_I = 0$ .

**Lemma 2.3.** *With the above notations, we have that, for any  $h \geq 1$ ,*

$$\deg f_* \mathcal{O}_X(h) = \sum_{i=0}^c \left( \sum_{|I|=i} (-1)^i \binom{h - k_I + r - 1}{r - 1} \frac{(h - k_I)d + y_I r}{r} \right) \quad (2.4)$$

Note that here we use the standard convention that considers equal to zero a binomial of the form  $\binom{n}{m}$  when  $n < m$ .

*Proof.* The Koszul sequence that provides the resolution of the ideal sheaf  $\mathcal{I}_{X|\mathbb{P}}(h)$  is the following:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-X_1 - \dots - X_c + hH) &\longrightarrow \bigoplus_{i=1}^c \mathcal{O}_{\mathbb{P}}(-X_1 - \dots - \widehat{X_i} \dots - X_c + hH) \longrightarrow \dots \\ \dots &\longrightarrow \bigoplus_{|I|=l} \mathcal{O}_{\mathbb{P}}(-X_I + hH) \longrightarrow \dots \longrightarrow \bigoplus_{i=1}^c \mathcal{O}_{\mathbb{P}}(-X_i + hH) \longrightarrow \mathcal{I}_{X|\mathbb{P}}(h) \longrightarrow 0, \end{aligned}$$

where the notation  $X_I$  means the divisor  $\sum_{j=1}^{|I|} X_{i_j}$ . Let us push forward this sequence by  $\pi_*$ : it remains exact because  $R^1\pi_*\mathcal{O}_{\mathbb{P}}(-X_1 - \dots - X_c + hH)$  vanishes for any  $h$ .

Observe now that for any integer  $a \geq 0$  and for any  $M$  divisor on  $B$  of degree  $y$ ,

$$\pi_*\mathcal{O}_{\mathbb{P}}(aH - M) \cong \pi_*\mathcal{O}_{\mathbb{P}}(a) \otimes \mathcal{O}_B(-M) \cong \text{Sym}^a \mathcal{E} \otimes \mathcal{O}_B(-M),$$

and that

$$\deg \text{Sym}^a \mathcal{E} \otimes \mathcal{O}_B(-M) = \binom{a+r-1}{r-1} \frac{ad - yr}{r}.$$

Notice moreover that for  $a < 0$  we have  $\pi_*\mathcal{O}_{\mathbb{P}}(a) = 0$ , and so the formula above still holds.

It is now immediate to compute  $\deg f_*\mathcal{O}_X(h) = \deg \pi_*\mathcal{O}_{\mathbb{P}}(h) - \deg \pi_*\mathcal{I}_{X|\mathbb{P}}(h)$  and obtain formula (2.4).  $\square$

*Proof. of Theorem 2.1.* Let  $H_X$  be the class of  $\mathcal{O}_X(1)$ ,  $F$  the class of a fibre of  $f$ , and  $H_F$  the class of  $\mathcal{O}_F(1)$ . Then we have the following formulas for the top intersection of these classes.

$$\begin{aligned} (hH_X)^{r-c} &= h^{r-c} H^{r-c} (k_1 H - M_1) \dots (k_c H - M_c) = \\ &= h^{r-c} \left( \prod_{i=1}^c k_i H^r - \sum_{i=1}^c (\prod_{j \neq i} k_j) y_i H^{r-1} \Sigma \right) = \\ &= h^{r-c} \left( \prod_{i=1}^c k_i d - \sum_{i=1}^c (\prod_{j \neq i} k_j) y_i \right); \\ H_F^{r-c-1} &= H^{r-c-1} (k_1 H - M_1) \dots (k_c H - M_c) \Sigma = \prod_{i=1}^c k_i H^{r-1} \Sigma = \prod_{i=1}^c k_i. \end{aligned}$$

Moreover, by using the Koszul sequence in Lemma 2.3, we have that

$$h^0(F, \mathcal{O}_F(h)) = \text{rank } f_*\mathcal{O}_X(h) = \sum_{i=0}^c \left( \sum_{|I|=i} (-1)^i \binom{h - k_I + r - 1}{r - 1} \right).$$

Combining this formula with the one of Lemma 2.3, we see that  $f$ -positivity of  $\mathcal{O}_X(h)$  is equivalent to the following inequality

$$\begin{aligned} h \left( \prod_{i=1}^c k_i d - \sum_{i=1}^c (\prod_{j \neq i} k_j) y_i \right) \sum_{i=0}^c \left( \sum_{|I|=i} (-1)^i \binom{h - k_I + r - 1}{r - 1} \right) \geq \\ (r - c) \prod_{i=1}^c k_i \left[ \sum_{i=0}^c \left( \sum_{|I|=i} (-1)^i \binom{h - k_I + r - 1}{r - 1} \frac{(h - k_I) d + y_I r}{r} \right) \right]. \end{aligned}$$

Grouping terms, this inequality becomes

$$\begin{aligned} \frac{h}{r} \sum_{i=0}^c \left( \sum_{|I|=i} (-1)^i \binom{h - k_I + r - 1}{r - 1} \right) \left[ c \prod_{i=1}^c k_i d - \sum_{i=1}^c (\prod_{j \neq i} k_j) y_i r \right] + \\ + (r - c) \prod_{i=1}^c k_i \left[ \sum_{i=0}^c \left( \sum_{|I|=i} (-1)^i \binom{h - k_I + r - 1}{r - 1} \frac{k_I d - y_I r}{r} \right) \right] \geq 0. \end{aligned} \tag{2.5}$$

Now observe that for  $h < \min_i \{k_i\}$ , then the second term in inequality (2.5) vanishes when  $i > 0$  because all binomials are zero, and trivially vanishes when  $i = 0$ . Hence in this case the inequality just becomes

$$\frac{h}{r} \binom{h+r-1}{r-1} \left( c \prod_{i=1}^c k_i d - \sum_{i=1}^c \left( \prod_{j \neq i} k_j \right) y_i r \right) \geq 0.$$

Moreover, the second term of (2.5) is  $\mathcal{O}(h^{r-1})$ , so the leading coefficient as a polynomial in  $h$  is a positive multiple of

$$\frac{1}{r} c \prod_{i=1}^c k_i d - \sum_{i=1}^c \left( \prod_{j \neq i} k_j \right) y_i r.$$

Thus the condition for  $f$ -positivity in both cases  $h < \min_i \{k_i\}$ , and  $h \gg 0$  is precisely

$$c \prod_{i=1}^c k_i d - \sum_{i=1}^c \left( \prod_{j \neq i} k_j \right) y_i r \geq 0 \iff \sum_{i=1}^c \frac{y_i}{k_i} \leq c \frac{d}{r} = c\mu$$

and the proof is concluded.  $\square$

**Remark 2.6.** It is worth recalling that  $f$ -positivity is stable under sum of pullbacks of divisors on the base curve. Indeed, if a divisor  $D$  on  $X$  is  $f$ -positive, then the divisor  $D + f^*M$  is  $f$ -positive for any divisor  $M$  on  $B$ .

**Remark 2.7.** In case  $c = 1$  both terms in inequality (2.5) are multiple of the number  $(dk_1 - ry_1)$  and one can see -as we do in [2]- that the multiplying term is positive, so that condition (1) of Theorem 2.1 can be substituted by asking that  $\mathcal{O}_X(h)$  being  $f$ -positive for *any* fixed value  $h \geq 1$ .

In the general case it seems not immediate to control the positivity of the whole quantity in inequality (2.5). Consider for instance the relative canonical divisor

$$K_f \equiv \left( \sum_{i=1}^c k_i - r \right) H_X - \left( \sum_{i=1}^c y_i - d \right) F.$$

From Theorem 2.1 it is difficult to give a general result that can ensure the  $f$ -positivity of  $K_f$  (i.e. whether the slope inequality is satisfied). One interesting case with affirmative answer is the one of codimension 2 complete intersection cycles with  $k_1, k_2 < r$ , as follows from Theorem 2.1.

### 3 The cones of cycles of $\mathbb{P}_B(\mathcal{E})$

In this section we interpret the result above in terms of cones in the Néron Severi space of codimension  $c$  cycles of  $X$ . We see that condition (1) of Theorem 2.1 defines a meaningful cone, which is intermediate with respect to the nef and the pseudoeffective ones. For stating this, we rewrite in a more compact form a result of Fulger [5], that describes completely this last two cones.

Recall first that given any vector bundle  $\mathcal{E}$  over a curve  $B$ , the Néron Severi space of codimension  $c$  cycles of  $\mathbb{P} = \mathbb{P}_B(\mathcal{E})$

$$N^c(X) := \frac{\{\text{The real span of classes of } c\text{-dimensional subvarieties of } X\}}{\text{numerical equivalence}}$$

is 2-dimensional, generated by the classes  $H^c$  and  $H^{c-1}\Sigma$ , where  $H = [\mathcal{O}_{\mathbb{P}}(1)]$  and  $\Sigma$  is the class of a fibre of  $\pi$ .

Let us consider the Harder-Narashiman filtration of  $\mathcal{E}$

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_l = \mathcal{E},$$

and call  $\mu_i := \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ , and  $\mu := \mu(\mathcal{E})$ . Recall in particular that

$$\mu_\ell < \mu_{\ell-1} < \dots < \mu_1, \tag{3.1}$$

and that  $\mu_\ell < \mu < \mu_1$  unless  $\mathcal{E}$  is semistable, in which case  $\mathcal{E}_1 = \mathcal{E}_\ell = \mathcal{E}$ .

It is useful to introduce the following notation.

**Definition 3.2.** With the above notations, we define the *virtual slopes*  $\tilde{\mu}_1 \leq \tilde{\mu}_2 \leq \dots \leq \tilde{\mu}_r$  of  $\mathcal{E}$  as follows. Let  $r_j$  be the rank of  $\mathcal{E}_j$  and let  $i \in \{1, \dots, r-1\}$ . If  $r_j \leq i < r_{j+1}$  then we define  $\tilde{\mu}_i = \mu_j$ . We define coherently  $\tilde{\mu}_r = \mu_\ell$ .

Observe that  $d = \deg \mathcal{E} = r\mu = \sum_{i=1}^r \tilde{\mu}_i$ . With this notation, we can restate Fulger's result as follows.

**Theorem 3.3** (Fulger [5]).

$$\text{Pseff}^c(\mathbb{P}) = \langle H^{c-1}\Sigma, H^c - (\tilde{\mu}_1 + \dots + \tilde{\mu}_c)H^{c-1}\Sigma \rangle; \quad (3.4)$$

$$\text{Nef}^c(\mathbb{P}) = \langle H^{c-1}\Sigma, H^c - (\tilde{\mu}_{r-c+1} + \dots + \tilde{\mu}_r)H^{c-1}\Sigma \rangle. \quad (3.5)$$

*Proof.* The first formula is just the content of Theorem 1.1 in [5] with the notation suitably adapted. Observe that in [5] the indexes of slopes are reversed with respect to our definition. With this in mind, and following the construction given by Figure 2 in [5], one immediately obtains that  $\nu^{(i)} = -(\tilde{\mu}_1 + \dots + \tilde{\mu}_c)$ . For the second formula just use the fact that  $\text{Nef}^c(\mathbb{P})$  is the subset of  $\text{Pseff}^c(\mathbb{P})$  defined by positivity product with

$$\text{Pseff}_c(\mathbb{P}) = \text{Pseff}^{r-c}(\mathbb{P}) = \langle H^{r-c-1}, H^{r-c} - (\tilde{\mu}_1 + \dots + \tilde{\mu}_{r-c})\Sigma \rangle,$$

and so it is the two dimensional cone determined by  $H^{c-1}\Sigma$  and  $H^c - aH^{c-1}\Sigma$  such that

$$\begin{aligned} 0 &= (H^c - aH^{c-1}\Sigma) (H^{r-c} - (\tilde{\mu}_1 + \dots + \tilde{\mu}_{r-c})\Sigma) = H^r - (a + \tilde{\mu}_1 + \dots + \tilde{\mu}_{r-c}) = \\ &= d - (a + \tilde{\mu}_1 + \dots + \tilde{\mu}_{r-c}) = \tilde{\mu}_{r-c+1} + \dots + \tilde{\mu}_r - a. \end{aligned}$$

□

We can now reformulate Theorem 2.1 using the language of cones. Let  $\mathbb{B}$  be the cone in  $\text{Nef}^c(\mathbb{P})$  generated by the classes  $[H^{c-1}\Sigma]$  and  $[H^c - c\mu H^{c-1}\Sigma]$ . Note that for any  $c \in \{1, \dots, r-1\}$  we have that

- $\sum_{i=1}^c \tilde{\mu}_i > c\mu$ ;
- $\sum_{i=1}^c \tilde{\mu}_{r-i+1} < c\mu$ .

This means that the cone  $\mathbb{B}$  is indeed contained in the pseudoeffective cone and contains the nef cone:

$$\text{Nef}^c(\mathbb{P}) \subseteq \mathbb{B} \subseteq \text{Pseff}^c(\mathbb{P}),$$

Note that the inclusions are strict unless  $\mathcal{E}$  is semistable (in which case the cones all coincide). Theorem 2.1 tells the following:

**Proposition 3.6.** *Let  $X \subset \mathbb{P}$  be a codimension  $c$  cycle which is a complete intersection of  $c$  relative hypersurfaces  $X_1, \dots, X_c$  in  $\mathbb{P}$  of degree at least 2. The numerical class of  $X$  is contained in  $\mathbb{B}$  if and only if the sheaf  $\mathcal{O}_X(1)$  is  $f$ -positive and if and only if it is asymptotically  $f$ -positive.*

## 4 $f$ -positivity and stability

In this section we derive some results on the  $f$ -positivity and the Chow stability of fibres of a fibration whose general fibres are complete intersections. For the definition of Chow stability of a projective variety see [4] and the references therein. From now on, anytime we say (semi)stable we mean Chow (semi)stable. The main result relating this conditions is due to Bost [3] (see [1] for references to similar results).

**Theorem 4.1** (Bost). *Let  $X$  be an  $n$ -dimensional variety with a flat proper surjective morphism  $f: X \rightarrow B$  over a smooth curve  $B$ . Let  $\mathcal{L}$  be a line bundle over  $X$  which is relatively ample with respect to  $f$ . If the general fibre of  $f$  is Chow semistable with respect to the immersion induced by the fibre of  $\mathcal{L}$  then  $\mathcal{L}$  is  $f$ -positive.*

What can we say about the stability of the fibres in our case? We now state a very natural stability result, which derives from a formula of R. G. Ferretti [4, Theorem 1.5]. This application of Ferretti's result was suggested to the second author by Yongnam Lee.

Let  $Y$  and  $Z$  be two irreducible subvarieties of  $\mathbb{P}^n = \mathbb{P}(V^\vee)$  whose intersection is proper. Let  $Y \cdot Z$  be the intersection cycle of  $Y$  and  $Z$ .

**Proposition 4.2.** *If  $Y$  and  $Z$  are semistable then  $Y \cdot Z$  is semistable. If, moreover, at least one among  $Y$  and  $Z$  is stable then the intersection  $Y \cdot Z$  is stable.*

*Proof.* We use the Hilbert-Mumford criterion for stability. Let us consider a 1-parameter subgroup of  $GL(V)$  and let  $F$  be the associated weighted filtration of  $V$ , with weights  $r_i$ . Then, for any subvariety  $X \subset \mathbb{P}^n$ , it is well defined an integer  $e_F(X)$  which in the notation of [4] is called degree of contact. The Hilbert-Mumford criterion says that an irreducible subvariety  $T \subset \mathbb{P}(V^\vee)$  is semistable (resp. stable) if and only if for any weighted filtration  $F$  of  $V$

$$\frac{e_F(T)}{(\dim T + 1)(\deg T)} \leq \frac{1}{n+1} \sum_{i=0}^n r_i \quad (\text{resp. } <). \quad (4.3)$$

Choosing  $Y$  and  $Z$  properly intersecting in  $\mathbb{P}(V^\vee)$ , Ferretti proves in [4, Theorem 1.5] the following “Bézout type” formula for the degree of contact of the cycle intersection  $Y \cdot Z$ :

$$e_F(Y \cdot Z) = \deg(Y)e_F(Z) + \deg(Z)e_F(Y) - \deg(Y)\deg(Z) \sum_{i=0}^n r_i. \quad (4.4)$$

Let us now suppose that  $Y$  and  $Z$  are semistable. From the Hilbert-Mumford criterion we have that

$$\frac{e_F(Y)}{(\dim Y + 1)(\deg Y)} \leq \frac{1}{n+1} \sum_{i=0}^n r_i, \quad \frac{e_F(Z)}{(\dim Z + 1)(\deg Z)} \leq \frac{1}{n+1} \sum_{i=0}^n r_i.$$

Call  $y$  and  $z$  the dimensions of  $Y$  and  $Z$  respectively. By properness assumption we have that

$$\dim(Y \cdot Z) = y + z - n.$$

We thus have the following chain of inequalities:

$$\begin{aligned} \frac{e_F(Y \cdot Z)}{(\dim(Y \cdot Z) + 1)(\deg(Y \cdot Z))} &= \frac{e_F(Y)}{(y + z - n + 1)\deg(Y)} + \frac{e_F(Z)}{(y + z - n + 1)\deg(Z)} \leq \\ &\leq \frac{(z + 1) + (y + 1)}{(y + z - n + 1)(n + 1)} \sum_{i=0}^n r_i - \frac{\sum_{i=0}^n r_i}{y + z - n + 1} = \\ &= \frac{1}{n + 1} \sum_{i=0}^n r_i, \end{aligned}$$

as wanted. The first equality is (4.4), the second inequality is the condition of semistability. The result with strict stability follows by substituting strict inequality for (at least) one of the varieties.  $\square$

**Theorem 4.5.** *Let  $X$  be an  $n$ -dimensional variety with a flat proper surjective morphism  $f: X \rightarrow B$  over a smooth curve  $B$ . Let  $\mathcal{L}$  be a line bundle over  $X$  which is relatively ample with respect to  $f$ . Suppose that*

*(\*) the general fibre  $F$  of  $f$  is embedded by  $\mathcal{L}|_F$  as a complete intersection of smooth hypersurfaces.*

*Then  $\mathcal{L}$  is  $f$ -positive.*

**Remark 4.6.** Using a result of Lee [6] we can weaken a lot the assumption of smoothness. Indeed, we could substitute condition (\*) with the following condition:

(\*\*) the general fibre  $F$  is embedded in  $\mathbb{P}^{h^0(F, \mathcal{L}|_F)-1} = \mathbb{P}^{r-1}$  by  $|\mathcal{L}|_F|$  as the complete intersection of  $r - n$  hypersurfaces  $Y_i$  of degree  $d_i$ , such that for any  $i = 1, \dots, r - n$

$$lct(\mathbb{P}^{r-1}, Y_i) \geq \frac{r}{d_i},$$

where  $lct$  is the log canonical threshold of the couple  $(\mathbb{P}^{r-1}, Y_i)$ .

**Remark 4.7.** Theorem 4.5 above is an extremely general result of  $f$ -positivity, its assumption (\*) being on the general fibres. However notice that in the case of a global complete intersection in  $\mathbb{P}$  it does not imply the results of Section 2.

We can on the other hand use Theorem 2.1 vice versa as in [2], and prove the following instability condition for the fibres of a global relative complete intersection. As usual, let  $\mathcal{E}$  be a rank  $r \geq 3$  vector bundle over a curve  $B$ . Let  $X \subset \mathbb{P}$  a relative complete intersection in the projective bundle  $\mathbb{P} = \mathbb{P}_B(\mathcal{E})$ , as in Section 2.

**Corollary 4.8.** *With the above notation, if  $\sum_{i=1}^c \frac{y_i}{k_i} > c\mu$  (equivalently  $[X] \notin \mathbb{B}$ ), then:*

- (i) *the fibres of  $f$  are Chow unstable with the restriction of  $\mathcal{O}_{\mathbb{P}^{r-1}}(h)$  for any  $h < \min\{k_i\}$ ;*
- (ii) *the fibres of  $f$  are Chow unstable with the restriction of  $\mathcal{O}_{\mathbb{P}^{r-1}}(h)$  for any  $h \gg 0$ .*

*Proof.* Immediate from Theorem 2.1 and Theorem 4.1. □

In [2], in the codimension one case, we proved a more general instability condition, and this led us to a singularity condition (a bound on the log canonical threshold of the fibres of  $f$  via Lee's result). In the general codimension case, it is not so easy to get geometric information from an unstability condition. It would be interesting to find an example with unstable and non-singular general fibre.

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